Achieving Long-term Fairness in Sequential Decision Making (Technical Appendix)

Yaowei Hu, Lu Zhang
University of Arkansas
{yaoweihu, lz006}@uark.edu

Proof of Lemma 2

Lemma 1. *For any t, suppose that $\mathbf{X}^{t+1}$ are c-sensitive, then distribution $P(\mathbf{X}^t|do(s_\pi,\theta))$ is ε-sensitive with $\varepsilon \leq 2mc(t-1)$, where m is the maximum ground distance between two values of $\mathbf{X}^t$.*

Proof. Let $D_{\mathbf{x}^t}(\theta)$ denote probability $P(\mathbf{x}^t|do(s_\pi,\theta))$ and $D(\theta)$ denote the corresponding distribution. We adopt a simple greedy strategy to solve the transportation problem to obtain a upper bound of $W_1(D(\theta),D(\theta'))$. We transverse through each value of $\mathbf{X}^t$. For each $\mathbf{x}^t$, if the amount of dirt in $D_{\mathbf{x}^t}(\theta)$ is larger than that of $D_{\mathbf{x}^t}(\theta')$, then we move the additional dirt to a pool. If the amount of dirt in $D_{\mathbf{x}^t}(\theta)$ is less than that of $D_{\mathbf{x}^t}(\theta')$, then we insert this demand into a queue and move the dirt from the pool to $D_{\mathbf{x}^t}(\theta)$ as soon as there is enough dirt in the pool. As a result, the total amount of dirt moved by this strategy is $\sum_{\mathbf{X}^t}|D_{\mathbf{x}^t}(\theta)-D_{\mathbf{x}^t}(\theta')|$. Thus, we have

$$W_1(D(\theta),D(\theta')) \leq \sum_{\mathbf{X}^t}|D_{\mathbf{x}^t}(\theta)-D_{\mathbf{x}^t}(\theta')|\cdot m, \quad (1)$$

where m is maximum ground distance between two values of $\mathbf{X}^t$. Then, according to the mean value theorem and Cauchy–Schwarz inequality, we have

$$\begin{aligned}|D_{\mathbf{x}^t}(\theta)-D_{\mathbf{x}^t}(\theta')| &= |\nabla D_{\mathbf{x}^t}(\eta)\cdot(\theta-\theta')| \\ &\leq \|\nabla D_{\mathbf{x}^t}(\eta)\|\|\theta-\theta'\|\end{aligned} \quad (2)$$

for some $\eta \in [\theta,\theta']$. By definition of $D_{\mathbf{x}^t}(\theta)$, it follows that

$$\begin{aligned}&D_{\mathbf{x}^t}(\theta) := P(\mathbf{x}^t|do(s_\pi,\theta)) \\ &= \sum_{\mathbf{X}^1,Y^1,\cdots,Y^{t-1}} P(\mathbf{x}^1|s)P_\theta(y^1|\mathbf{x}^1,s)\cdots P(\mathbf{x}^t|x^{t-1},y^{t-1}).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\nabla D_{\mathbf{x}^t}(\theta) = \sum_{\mathbf{X}^1,Y^1,\cdots,Y^{t-1}} \{P(\mathbf{x}^1|s)\nabla P_\theta(y^1|\mathbf{x}^1,s)\cdots P(\mathbf{x}^t|x^{t-1},y^{t-1}) \\ &+ P(\mathbf{x}^1|s)P_\theta(y^1|\mathbf{x}^1,s)P(\mathbf{x}^2|\mathbf{x}^1,y^1)\nabla P_\theta(y^2|\mathbf{x}^2,s)\cdots \\ &+\cdots\}\end{aligned}$$

According to the definition of c-sensitivity, we have

$$\|\sum_{Y^t}\nabla_\theta P_\theta(y^t|\mathbf{x}^t,s)P(\mathbf{x}^{t+1}|\mathbf{x}^t,y^t)\| \leq c\sum_{Y^t}P(\mathbf{x}^{t+1}|\mathbf{x}^t,y^t).$$

By the triangle inequality, it follows that

$$\begin{aligned}&\|\nabla D_{\mathbf{x}^t}(\theta)\| \leq \sum_{\mathbf{X}^1,Y^1,\cdots,Y^{t-1}} \{P(\mathbf{x}^1|s)cP(\mathbf{x}^2|\mathbf{x}^1,y^1)\cdots P(\mathbf{x}^t|x^{t-1},y^{t-1}) \\ &+ P(\mathbf{x}^1|s)P_\theta(y^1|\mathbf{x}^1,s)P(\mathbf{x}^2|\mathbf{x}^1,y^1)cP(\mathbf{x}^3|\mathbf{x}^2,y^2)\cdots \\ &+\cdots\} \\ &= c\sum_{\mathbf{X}^1,Y^1,\cdots,Y^{t-1}} \{P(\mathbf{x}^1,\mathbf{x}^2,\cdots,\mathbf{x}^t|do(y^1)) \\ &+P(\mathbf{x}^1,y^1,\cdots,\mathbf{x}^t|do(y^2))+\cdots\} \\ &= c\left\{\sum_{Y^1}P_\theta(\mathbf{x}^t|do(s,y^1))+\cdots+\sum_{Y^{t-1}}P_\theta(\mathbf{x}^t|do(s,y^{t-1}))\right\},\end{aligned} \quad (3)$$

where the second step is based on the truncated factorization formula of computing the do-operation. Combining Eqs. (1), (2), and (3), we have

$$\begin{aligned}W_1(D(\theta),D(\theta')) &\leq mc\sum_{\mathbf{X}}\left\{\sum_{Y^1}P_\eta(\mathbf{x}^t|do(s,y^1))+ \right. \\ &\left.\cdots+\sum_{Y^{t-1}}P_\eta(\mathbf{x}^t|do(s,y^{t-1}))\right\}\|\theta-\theta'\| \\ &= 2mc(t-1)\|\theta-\theta'\|.\end{aligned}$$

Hence, the lemma is proven. □

Proof of Theorem 1

Theorem 1. *Suppose that surrogated loss function $(\phi\circ h)(\cdot)$ is β–jointly smooth and γ-strongly convex, and suppose that $\mathbf{X}^{t+1}$ are c-sensitive for any t, then the repeated risk minimization converges to a stable point at a linear rate, if $2mc(t^*-1) < \frac{\beta}{\gamma}$.*

Proof. This proof basically follows the proof of Theorem 3.5 in (Perdomo et al. 2020).

Fix $\theta, \theta' \in \Theta$. Let

$$f_a(\varphi) = \sum_{t=1}^{t^*} \mathop{\mathbb{E}}_{S, \mathbf{X}^t, Y^t \sim P(S, \mathbf{X}^t, Y^t)} \left[\phi\left(Y^t h_\varphi(\mathbf{X}^t, S)\right)\right],$$

$$f_l(\varphi) = \frac{1}{2}\left\{\mathop{\mathbb{E}}_{\mathbf{X}^{t^*} \sim P(\mathbf{X}^{t^*}|do((s_\pi^+, \theta))} \left[\phi\left(-h_\varphi\left(\mathbf{X}^{t^*}, s^-\right)\right)\right] + \mathop{\mathbb{E}}_{\mathbf{X}^{t^*} \sim P(\mathbf{X}^{t^*}|do((s_\pi^-, \theta))} \left[\phi\left(h_\varphi\left(\mathbf{X}^{t^*}, s^-\right)\right)\right] - 1\right\},$$

$$f_s(\varphi) = \frac{1}{t^*}\sum_{t=1}^{t^*}\left\{\mathop{\mathbb{E}}_{\mathbf{X}^{t} \sim P(\mathbf{X}^{t}|do((s_{\pi^t}^+, \theta))} \left[\phi\left(-h_\varphi\left(\mathbf{X}^{t^*}, s^-\right)\right)\right] + \mathop{\mathbb{E}}_{\mathbf{X}^{t^*} \sim P(\mathbf{X}^{t^*}|do((s_{\pi^t}^-, \theta))} \left[\phi\left(h_\varphi\left(\mathbf{X}^{t^*}, s^-\right)\right)\right] - 1\right\},$$

and

$$f(\varphi) = \lambda_a f_a(\varphi) + \lambda_l f_l(\varphi) + \lambda_s f_s(\varphi).$$

Define $f'(\varphi)$ similarly to $f(\varphi)$ by replacing θ with θ'. Let $G(\theta) = \mathrm{argmin}_\varphi f(\varphi)$. Since $(\phi \circ h)(\cdot)$ is γ-strongly convex, $f(\cdot)$ is at least γ-strongly convex. Then, we have

$$f(G(\theta)) - f(G(\theta')) \geq (G(\theta) - G(\theta)')^\top \nabla f(G(\theta')) + \frac{\gamma}{2}\|G(\theta) - G(\theta')\|_2^2,$$

$$f(G(\theta')) - f(G(\theta)) \geq \frac{\gamma}{2}\|G(\theta) - G(\theta')\|_2^2.$$

Combining the two inequalities we have

$$-\gamma\|G(\theta) - G(\theta')\|_2^2 \geq (G(\theta) - G(\theta)')^\top \nabla f(G(\theta')). \quad (4)$$

On the other hand, since $(\phi \circ h)(\cdot)$ is β-jointly smooth, by applying Cauchy-Schwarz inequality we have that $(G(\theta) - G(\theta)')^\top \nabla\phi(h_{G(\theta')}(\mathbf{x}^{t^*}, s))$ is $\|G(\theta) - G(\theta')\|_2\beta$-Lipschitz. Using the dual formulation of the optimal transport distance and Lemma 1, we have

$$(G(\theta) - G(\theta)')^\top \nabla f_l(G(\theta')) - (G(\theta) - G(\theta)')^\top \nabla f_l'(G(\theta')) \geq -\varepsilon\beta\|G(\theta) - G(\theta')\|_2\|\theta - \theta'\|_2,$$

$$(G(\theta) - G(\theta)')^\top \nabla f_s(G(\theta')) - (G(\theta) - G(\theta)')^\top \nabla f_s'(G(\theta')) \geq -\varepsilon\beta\|G(\theta) - G(\theta')\|_2\|\theta - \theta'\|_2,$$

where $\varepsilon = 2mc(t^* - 1)$. In addition, we have

$$(G(\theta) - G(\theta)')^\top \nabla f_a(G(\theta')) - (G(\theta) - G(\theta)')^\top \nabla f_a'(G(\theta')) = 0$$

Adding up above three (in)equalities, we have

$$(G(\theta) - G(\theta)')^\top \nabla f(G(\theta')) - (G(\theta) - G(\theta)')^\top \nabla f'(G(\theta')) \geq -\varepsilon\beta\|G(\theta) - G(\theta')\|_2\|\theta - \theta'\|_2.$$

Due to the first-order optimality conditions for convex functions, it follows that

$$(G(\theta) - G(\theta)')^\top \nabla f(G(\theta')) \geq -\varepsilon\beta\|G(\theta) - G(\theta')\|_2\|\theta - \theta'\|_2. \quad (5)$$

Combining Eqs. (4) and (5), we have

$$-\gamma\|G(\theta) - G(\theta')\|_2^2 \geq -\varepsilon\beta\|G(\theta) - G(\theta')\|_2\|\theta - \theta'\|_2.$$

By rearranging, we have

$$\|G(\theta) - G(\theta')\|_2 \leq \varepsilon\frac{\beta}{\gamma}\|\theta - \theta'\|_2.$$

Let θ_{PS} be a stable point, i.e., $G(\theta_{\mathrm{PS}}) = \theta_{\mathrm{PS}}$. In addition, by definition we have $\theta_i = G(\theta_{i-1})$. Thus, it follows that

$$\|\theta_i - \theta_{\mathrm{PS}}\| \leq \varepsilon\frac{\beta}{\gamma}\|\theta_{i-1} - \theta_{\mathrm{PS}}\|_2 \leq \left(\varepsilon\frac{\beta}{\gamma}\right)^i\|\theta_0 - \theta_{\mathrm{PS}}\|_2.$$

Therefore, if $\varepsilon = 2mc(t^* - 1) < \frac{\beta}{\gamma}$, the RRM converge to θ_{PS} at a linear rate.

Hence, the theorem is proven.

□

References

Perdomo, J.; Zrnic, T.; Mendler-Dünner, C.; and Hardt, M. 2020. Performative prediction. In *International Conference on Machine Learning*, 7599–7609. PMLR.